ON SOME TYPES OF FLOW OF A CONDUCTING FLUID IN PIPES OF RECTANGULAR CROSS-SECTION, PLACED IN A MAGNETIC FIELD

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We examine the solution of the integral equations for the problem of the steady flow of a conducting fluid in a rectangular pipe with two non-conducting walls and two conducting walls, the latter in planes parallel to the outside magnetic field. Also, we give a new form of the solution of Shercliff's [1] problem of flow in a pipe with non-conducting walls. The actual solutions given are valid, in particular, also for large values of Hartmann number.

1. We shall consider the steady flow of a conducting fluid in a pipe of rectangular cross-section with two conducting walls in planes parallel to the exterior magnetic field H°, the direction of which we take along the x-axis.

The stated problem was already considered in [2], where it was assumed that the velocity field and the electric and magnetic fields in a fluid do not depend on the z-coordinate along the axis of the pipe, that the pressure gradient $\partial p/\partial z = P$ is constant over the length as well as the cross-section of the pipe and that a current I per unit length of pipe flows through the ideally conducting walls of the pipe.

The problem was reduced to the solution of the integral equation

$$\int_0^l [G(\xi, d, x, 0) + G(\xi, 0, x, 0)]\, f_0(\xi) \cosh\gamma(x-\xi)\, d\xi = \frac{2\alpha \sinh\gamma x \sinh\gamma(l-x)}{\sinh\gamma l} \qquad (0 \leqslant x \leqslant l) \tag{1.1}$$

where

$$G(\xi, \eta, x, y) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \{K_0(\gamma\sqrt{(x_m-\xi)^2 + (y_n-\eta)^2}) +$$

$$+ K_0(\gamma \sqrt{(x_m - \xi)^2 + (y_n' - \eta)^2}) - K_0(\gamma \sqrt{(x_m' - \xi)^2 + (y_n' - \eta)^2}) -$$

$$- K_0(\gamma \sqrt{(x_m' - \xi)^2 + (y_n - \eta)^2}\}) \qquad \binom{0 \leqslant x \leqslant l}{0 \leqslant y \leqslant d} \tag{1.2}$$

where $K_0(z)$ is the McDonald function, l and d are the lengths of the sides of the rectangular cross-section of the pipe in the direction of the field $\mathbf{H}^\circ$ and perpendicular to it, respectively

$$x_m = 2ml + x, \quad x_m' = 2ml - x, \quad y_n = 2nd + y, \quad y_n' = 2nd - y \tag{1.3}$$

$$\gamma = \frac{\mu H^\circ}{2c}\sqrt{\frac{\sigma}{\eta}}, \qquad \alpha = \frac{1}{4\gamma\eta}\left(Pl - \frac{H^\circ \mu I}{c}\right) \tag{1.4}$$

where c is the velocity of light, σ, μ and η are the conductivity, magnetic permeability and viscosity of the fluid.

If $\gamma d \gg 1$ and $\gamma l \gg 1$, then Equation (1.1) is replaced by the following approximate equation*:

$$\int_0^1 \frac{\chi(\zeta)\, d\zeta}{\sqrt{|z - \zeta|}} = \frac{2 \sinh Mz \sinh M(1-z)}{\sinh M} + e^{-2Mz}\Phi(z) + e^{-2M(1-z)}\Psi(z) \tag{1.5}$$

where

$$\Phi(z) = \int_z^1 \frac{\chi(\zeta)\, d\zeta}{\sqrt{z + \zeta}}, \qquad \Psi(z) = \int_0^z \frac{\chi(\zeta)\, d\zeta}{\sqrt{2 - z - \zeta}}$$

$$z = \frac{x}{l}, \qquad f_0(x) = \frac{2\alpha\sqrt{2\pi M}}{l}\chi\left(\frac{x}{l}\right) \qquad (M = \gamma l \text{ is the Hartmann number})$$

2. We write Equation (1.5) in the form

$$\int_0^1 \frac{\chi(\zeta)\, d\zeta}{\sqrt{|z - \zeta|}} = w(z) \qquad (0 \leqslant z \leqslant 1) \tag{2.1}$$

where the following expression is obtained for $\chi(z)$ [3]:

$$\chi(z) = -\frac{1}{\sqrt{2\pi}\,\Gamma^2(3/4)}\frac{1}{z^{1/4}}\frac{d}{dz}\int_z^1 \frac{\sqrt{\zeta}\, d\zeta}{(\zeta - z)^{1/4}}\frac{d}{d\zeta}\int_0^\zeta \frac{w(\sigma)\, d\sigma}{\sigma^{1/4}(\zeta - \sigma)^{1/4}} \tag{2.2}$$

Equation (2.2) is solved by the method of successive approximations.

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* Compare the derivation of Equations (2.43) and (4.9) of [2].

As a first approximation (to which we confine ourselves here) we shall take an approximation in which in Equation (1.5) the slowly varying functions $\Phi(z)$ and $\Psi(z)$, which enter as factors of the rather rapidly varying functions e^{-2Mz} and $e^{-2M(1-z)}$, are replaced by their common value*

$$\int_0^1 \frac{\chi(\zeta)\,d\zeta}{\sqrt{\zeta}} = \int_0^1 \frac{\chi(\zeta)\,d\zeta}{\sqrt{1-\zeta}} \equiv A = \text{const} \tag{2.3}$$

for $z = 0$ and $z = 1$, i.e. by their value at the points where the rapidly decreasing exponents have their maximum value, equal to unity. If we neglect e^{-2M} as small with respect to unity**, then the function $w(z)$ becomes equal to

$$w(z) = 1 + (A - 1)\left(e^{-2Mz} + e^{-2M(1-z)}\right) \tag{2.4}$$

and $\chi(z)$ may be found from Equation (2.2), where the constant A must be determined from the condition $\chi(0) = 0$.

We assume that $\chi(z) = \chi_1(z) + (A - 1)[\chi_2(z) + \chi_3(z)]$. Also, we define the functions $\chi_i(z)$ by the equations

$$\int_0^1 \frac{\chi_1(\zeta)\,d\zeta}{\sqrt{|z-\zeta|}} = 1, \qquad \int_0^1 \frac{\chi_2(\zeta)\,d\zeta}{\sqrt{|z-\zeta|}} = e^{-2Mz}, \qquad \int_0^1 \frac{\chi_3(\zeta)\,d\zeta}{\sqrt{|z-\zeta|}} = e^{-2M(1-z)} \tag{2.5}$$

When replacing in the second equation z by $(1 - z)$ and ζ by $(1 - \zeta)$, we see that

$$\int_0^1 \frac{\chi_2(1-\zeta)\,d\zeta}{\sqrt{|z-\zeta|}} = e^{-2M(1-z)} \quad \text{or} \quad \chi_3(\zeta) = \chi_2(1-\zeta) \tag{2.6}$$

Thus it remains to find the solutions of the first and second equations of (2.5). This may be accomplished with the help of Formula (2.2). Omitting the quite lengthy calculations and transformations, we obtain

$$\chi_1(z) = \frac{1}{\pi\sqrt{2}\,[z(1-z)]^{1/4}} \tag{2.7}$$

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* Since $\chi(1 - z) = \chi(z)$.

** Note that the chosen function $w(z)$ satisfies with the same accuracy the relationship

$$w(0) = \int_0^1 \frac{\chi(\zeta)\,d\zeta}{\sqrt{\zeta}} = A$$

resulting from Equation (2.1) for $z = 0$.

$$\chi_2(z) = -\frac{1}{4\sqrt{2\pi}\,\Gamma(^3/_4)\,(2M)^{^1/_4}}\frac{1}{[z(1-z)]^{^1/_4}} + \qquad (v_m = [(1-z)(1-\sigma)]^{^1/_4})$$

$$+\frac{\sqrt{2}}{\sqrt{\pi}\,\Gamma^2(^3/_4)\,z^{^1/_4}}\int_0^1 \frac{e^{-2M\sigma}[6M\sigma - 8M^2\sigma^2]\,d\sigma}{\sigma^{^1/_4}}\int_0^{v_m}\frac{v^2dv}{\sqrt{(z-\sigma)^2+4z\sigma v^4}} \qquad (2.8)$$

where, again, in the last equation some of the terms of the order e^{-2M} have been dropped.

For $z = 0$ the double integral of (2.8) becomes equal to

$$\int_0^1 \frac{e^{-2M\sigma}[6M - 8M^2\sigma](1-\sigma)^{^3/_4}}{3\sigma^{^1/_4}}\,d\sigma \approx \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\sqrt{2M}\left(1+\frac{3}{32M}\right)$$

while for $z = 1$ it becomes zero. The condition

$$\chi(0) = \{\chi_1(z) + (A-1)[\chi_2(z) + \chi_3(z)]\}_{z\to 0} =$$
$$= \{\chi_1(z) + (A-1)[\chi_2(z) + \chi_2(1-z)]\}_{z\to 0} = 0$$

therefore gives upon elimination of the small quantities of order $M^{-5/4}$ the following equation

$$A - 1 = -\frac{\Gamma(^3/_4)}{(2M)^{^1/_4}\sqrt{\pi}} \qquad (2.9)$$

In this manner, the final formula for $\chi(z)$ with the assumed approximation has the form

$$\chi(z) = \frac{1}{\pi\sqrt{2}}\left\{\frac{(1+{}^1/_4 M)}{[z(1-z)]^{^1/_4}} - \frac{2}{(2M)^{^1/_4}\Gamma(^3/_4)}[F(z) + F(1-z)]\right\} \qquad (2.10)$$

Here

$$F(z) = \frac{1}{z^{^1/_4}}\int_0^1 \frac{e^{-2M\sigma}[6M\sigma - 8(M\sigma)^2]\,d\sigma}{\sigma^{^1/_4}}\int_0^{v_m}\frac{v^2dv}{\sqrt{(z-\sigma)^2+4z\sigma v^4}} \qquad (2.11)$$
$$(v_m = [(1-z)(1-\sigma)]^{^1/_4})$$

Note that the inner integral of this equation is expressible in terms of elliptic integrals of the first and second kind with modulus $k = 1/\sqrt{2}$, namely

$$\int_0^{v_m}\frac{v^2dv}{\sqrt{(z-\sigma)^2+4z\sigma v^4}} = \frac{\sqrt{\sin n\beta}}{2\sqrt{z\sigma}}\left\{\frac{1}{2}F\left(\varphi_m, \frac{1}{\sqrt{2}}\right) - E\left(\varphi_m, \frac{1}{\sqrt{2}}\right) + \right.$$
$$\left. + \frac{\tan\varphi_m/2}{1+\tan^2\varphi_m/2}\sqrt{1+\tan^4\frac{\varphi_m}{2}}\right\} \qquad (2.12)$$

where it is assumed that $z > \sigma$ and where the following notation has been introduced

$$\sqrt{\frac{z}{\sigma}} = e^{\beta}, \qquad \tan\frac{\varphi_m}{2} = \frac{v_m}{\sqrt{\sinh\beta}} \tag{2.13}$$

For $\sigma > z$ in (2.12) and (2.13) the places of z and σ must be interchanged. Equations (2.12) to (2.13) are valid for any $v_m \leqslant 1$, not only for $v_m = [(1 - z)(1 - \sigma)]^{1/4}$.

The validity of these equations is easily established, if we assume in the resulting integral

$$v = \left(\frac{(z-\sigma)^2}{4z\sigma}\right)^{1/4} \tan\frac{\varphi}{2} \equiv \sqrt{\sinh\beta}\tan\frac{\varphi}{2}$$

and write the obtained result in the following form

$$I = \frac{\sqrt{\sinh\beta}}{2\sqrt{z\sigma}} \lim_{\varphi_0\to 0}\left\{\int_{\varphi_0}^{\varphi_m} \frac{d\varphi}{\sin^2\varphi\sqrt{1-{}^1/_2\sin^2\varphi}} - \int_{\varphi_0}^{\varphi_m} \frac{\cos\varphi\, d\varphi}{\sin^2\varphi\sqrt{1-{}^1/_2\sin^2\varphi}} - \right.$$
$$\left. -\frac{1}{2}\int_{\varphi_0}^{\varphi_m} \frac{d\varphi}{\sqrt{1-{}^1/_2\sin^2\varphi}}\right\}$$

We use the relationship

$$\int\frac{d\varphi}{\sin^2\varphi\Delta(\varphi)} = \int\frac{d\varphi}{\Delta(\varphi)} - \int\Delta(\varphi)\,d\varphi - \cot\varphi\Delta(\varphi) + \mathrm{const}, \qquad \Delta(\varphi) = \sqrt{1-\tfrac{1}{2}\sin^2\varphi}$$

which is easily verified by differentiation.

3. We shall now investigate the obtained solution. First of all we note that the function $F(z)$ is positive in the neighborhood of $z = 0$, and then when z increases it changes sign and continues to be negative up to the value $z = 1$, where it becomes zero. This follows directly from Equation (2.11) if we notice that, because of the fast convergence of the integral over σ contained therein the only noticeable contribution to the value of $F(z)$ is made in the region in which σ is of the order less than $O(M^{-1})$.

In view of this we may assume for z substantially greater than M^{-1} in the inner integral of (2.11) that

$$\sqrt{(z-\sigma)^2 + 4z\sigma v^4} \approx z \tag{3.1}$$

As a result we obtain

$$\int_0^{v_m} \frac{v^2 dv}{\sqrt{(z-\sigma)^2 + 4z\sigma v^4}} \approx \frac{(1-z)^{3/4}}{3z} \qquad (v_m = [(1-z)(1-\sigma)]^{1/4}) \tag{3.2}$$

For the corresponding values of z this yields

$$F(z) \approx \frac{2(1-z)^{3/4}}{3z^{7/4}(2M)^{7/4}} \int_0^\infty e^{-t}\left[\frac{3}{2}t^{3/4} - t^{7/4}\right]dt = -\frac{\Gamma(3/4)(1-z)^{3/4}}{8(2M)^{7/4}z^{7/4}} \tag{3.3}$$

Using this formula, we obtain from (2.10) an approximate expression for $\chi(z)$, valid in the middle part of the interval $0 \leqslant z \leqslant 1$, i.e. at distances from its limits which are appreciably greater than M^{-1}

$$\chi(z) \approx \frac{1}{\pi\sqrt{2}\,[z(1-z)]^{1/4}}\left\{1 + \frac{1}{8Mz(1-z)}\right\} \tag{3.4}$$

Hence it is seen that in this region for sufficiently large M the function $\chi(z)$ differs little from $\chi_1(z)$. Because for such values of M Equation (3.4) must be valid in the greater part of the interval $(0 \leqslant z \leqslant 1)$, in addition to the narrow zones immediately near $z = 0$ and $z = 1$ and because at the boundaries of these zones $\chi(z)$ must become vary large, it follows that a sharp increase of the function $\chi(z)$ must take place inside these zones from the value it has on the outside boundaries of these zones, namely zero for $z = 0$ or for $z = 1$.

The behavior of function $\chi(z)$ in both these zones is the same (see Equation (2.10), from which it is seen that $\chi(1 - z) = \chi(z)$). Therefore, we shall investigate the behavior of $\chi(z)$ only in the vicinity $z = 0$. In Equations (2.10) and (2.11) we may assume

$$1 - z \approx 1, \qquad [(1-z)(1-\sigma)]^{1/4} \approx 1, \qquad F(1-z) \approx 0$$

and rewrite the expression for $\chi(z)$ as follows

$$\frac{\pi\sqrt{2}}{(2M)^{1/4}}\chi(z) \approx \frac{1}{u^{1/4}}\left\{1 - \frac{2}{\Gamma(3/4)}\int_0^\infty \frac{e^{-t}\,[3t - 2t^2]\,dt}{t^{1/4}} \int_0^1 \frac{v^2 dv}{\sqrt{(u-t)^2 + 4utv^4}}\right\} \equiv \omega(u)$$

$$u = 2Mz \tag{3.5}$$

The single-parametric function $\omega(u)$ may be tabulated* and, in particular the value $u_m = 2Mz_m$ may be found which corresponds to the maximum of the function $\chi(z)$, namely the value for which $\omega(u_m) = 0$.

Values of $\omega(u)$ for several values of u are tabulated below.

$u =$	1.0	1.5	2.0	3.0	3.5	4.0
$\omega(u) =$	0.798	0.868	0.872	0.840		0.794
$\Omega(u) =$	0.825	0.849	0.840	0.792	0.766	

* **Using Formulas (2.12) and (2.13) where it must be assumed that $v_m = 1$.**

According to the calculations

$$u_m \approx 1.7, \qquad \omega_{\max} = \omega(u_m) \approx 0.875$$

corresponding to these the function $\chi(z)$ has a maximum in the region near the wall

$$\chi_{\max} = \chi(z_m) = \frac{(2M)^{1/4}}{\pi\sqrt{2}}\,\omega_{\max} \approx 0.268M^{1/4} \quad \text{for} \quad z_m = \frac{u_m}{2M} \approx \frac{0.85}{M}$$

Note. For order of magnitude calculations it is useful to notice that for the integral

$$A(u, t) = \int_0^1 \frac{v^2 dv}{\sqrt{(u-t)^2 + 4utv^4}} \tag{3.6}$$

simple evaluations may be indicated at the upper and lower end, namely

$$\frac{1}{2u} \geqslant A(u, t) \geqslant \frac{1}{3u} \quad \text{for } u \geqslant t \geqslant 0 \tag{3.7}$$

where the upper limit is reached for $u = t$, and the lower for $t = 0$. For $t \geqslant u \geqslant 0$ the places of u and t in (3.7) must be interchanged.

The proof of the evaluations (3.7), and also of the related and more general treatment of the integral

$$I = \int_0^{v_m \leqslant 1} \frac{v^2 dv}{\sqrt{(u-t)^2 + 4utv^4}} \tag{3.8}$$

directly follows from the inequalities

$$I_1 = \int_0^{v_m} \frac{v^2 dv}{\sqrt{(u-t)^2 v^2 + 4utv^4}} \geqslant I = \int_0^{v_m} \frac{v^2 dv}{\sqrt{(u-t)^2 + 4utv^4}} \geqslant$$

$$\geqslant \int_0^{v_m} \frac{v^2 dv}{\sqrt{(u-t)^2 + 4utv^3}} = I_2$$

in which both limiting values of the integral may be computed and give

$$I_1 = \frac{1}{4ut}\left[\sqrt{(u-t)^2 + 4utv_m^2} - |u-t|\right] \tag{3.9}$$

$$I_2 = \frac{1}{6ut}\left[\sqrt{(u-t)^2 + 4utv_m^4} - |u-t|\right] \tag{3.10}$$

In the case $v_m = 1$, the relationships (3.7) are obtained, which show that, for example, for $u \geqslant t \geqslant 0$

$$A(u, t) = \frac{\theta(u, t)}{u}$$

where

$$\frac{1}{3} < (t, u) \leqslant \frac{1}{2}$$

i.e. $\theta(u, t)$ is a slowly variable function of u and t.

If we now assume that approximately

$$A(u, t) = \begin{cases} \frac{1}{3u} & \text{for } u \geqslant t \geqslant 0 \\ \frac{1}{3t} & \text{for } 0 \leqslant u \leqslant t \end{cases} \qquad (3.11)$$

and substitute these values in the middle portion of the Equation (3.5), then we obtain an approximate expression $\Omega(u)$ of the function $\omega(u)$ in the form

$$\Omega(u) = \frac{1}{u^{1/4}}\left\{1 - \frac{2}{\Gamma(3/4)}\left[\frac{1}{u}\int_0^u e^{-t}\left(t^{3/4} - \frac{2}{3}t^{7/4}\right)dt + \int_u^\infty e^{-t}\left(t^{-1/4} - \frac{2}{3}t^{3/4}\right)dt\right]\right\} =$$
$$= \frac{2}{u^{1/4}\Gamma(3/4)}\left\{\Gamma\left(\frac{3}{4}, u\right) - \frac{2}{3}\Gamma\left(\frac{7}{4}, u\right) - \frac{1}{u}\left[\Gamma\left(\frac{7}{4}, u\right) - \frac{2}{3}\Gamma\left(\frac{11}{4}, u\right)\right]\right\} \qquad (3.12)$$

where

$$\Gamma(n, u) = \int^u e^{-t} t^{n-1}\, dt$$

is an incomplete gamma* function. The values of the function $\Omega(u)$ for $0 \leqslant u \leqslant 3.5$ are for comparison given in the table on page 111. The maximum of $\Omega(u)$, which is 0.849, occurs at $u = U_m \approx 1.5$. To this corresponds in the given approximation the value

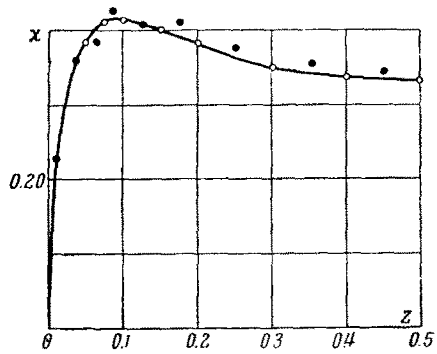
$$\chi_{\max} \approx \frac{\sqrt[4]{2M}}{\pi\sqrt{2}}\,\Omega(U_m) = 0.228\,M^{1/4} \qquad \text{for } Z \approx \frac{0.75}{M}$$

From the given data it is seen, that the discrepancy between corresponding values of $\omega(u)$ and $\Omega(u)$ does not exceed a few percent. The calculation of the values of the function $\omega(u)$, however, requires considerably larger expenditure of computing labor, than does the determination of values of $\Omega(u)$ by means of Equation (3.12).

In conclusion of this section in the figure a comparison is given of the results of the numerical solution of Equation (3.2) of [2] for a Hartmann number $M = 10$ with the results, obtainable for the same value of M using the formulas (2.10), (2.11) and (3.4), (3.5). In the case of numerical solution the following values of ζ_i in the Formula (3.3) of [2] were chosen: 0, 0.0125, 0.0375, 0.0625, 0.0875, 0.125, 0.175, 0.25, 0.35, 0.45 and the correspondingly located values in the other half of

* Tables of these functions exist. See, for example, [4,5].

the interval. The corresponding values of $\chi(\zeta)$ are represented in the figure by the solid points.



The values of $\chi(\zeta)$, obtained from (2.10), (2.11) and (3.4), (3.5) are represented by the circles through which the curve is drawn on the figure. The results obtained by both methods are in close agreement so that, even in the case of this comparatively low Hartmann number $M = 10$, the asymptotic form of the solution (2.10), (2.11) may serve as a good approximation to the exact solution of the problem.

4. When the function $\chi(z)$ is found, the solution of the problem amy be considered essentially completed; indeed, the velocities of the flow u and v, and also the second magnetic field H_z are found from Equations (2.34) and (2.35) of [2]. In these equations we must choose $d = \infty$ and as a result to eliminate $G(\xi, d, x, y)$, replace $G(\xi, 0, x, y)$ by $g(\xi, 0, x, y)$ and take into account that

$$f_0(\xi) = \frac{2\alpha}{l}\sqrt{2\pi M}\,\chi\left(\frac{\xi}{l}\right) \tag{4.1}$$

As a result we obtain

$$u(x, y) = \frac{1}{2\gamma}\left[\frac{H^\circ\mu}{4\pi\eta}H_z + \frac{P(x-a)}{\eta}\right] =$$
$$= \alpha\left\{\frac{2\sqrt{2\pi M}}{l}\int_0^l g(\xi, 0, x, y)\,\chi\left(\frac{\xi}{l}\right)\sinh\frac{M(x-\xi)}{l}\,d\xi - \frac{\sinh M(1-2x/l)}{\sinh M}\right\} \tag{4.2}$$

$$v(x, y) = -\alpha\left\{\frac{2\sqrt{2\pi M}}{l}\int_0^l g(\xi, 0, x, y)\,\chi\left(\frac{\xi}{l}\right)\operatorname{ch} M\,\frac{(x-\xi)}{l}\,d\xi - \right.$$
$$\left. - 2\,\frac{\sinh Mx/\ \sinh M(1-x/l)}{\sinh M}\right\} \tag{4.3}$$

Without going into a detailed analysis of these formulas, we shall note only that, in the case of large M, the integral terms in these equations decrease very rapidly with distance from the wall $y = 0$. Indeed, from Equation (2.42) of [2] it is seen that $g(\xi, 0, x, y)$ is composed of terms of the form

$$\frac{1}{2\pi}K_0\left[\frac{M}{l}\sqrt{(2ml \pm x - \xi)^2 + y^2}\right] \qquad (m = 0, \pm 1, \pm 2, \ldots)$$

which for $My/l >> 1$ asymptotically approach the values

$$\frac{1}{2}\sqrt{\frac{l}{2\pi M\sqrt{(2ml\pm x-\xi)^2+y^3}}}\exp\left(-\frac{M}{l}\sqrt{(2ml\pm x-\xi)^2+y^2}\right)\equiv J_m{}^{\pm}$$

so that, with increase of y, a rapid decrease occurs also in the functions $J_m \pm \cosh M(x - \xi)/l$ and $J_m \pm \sinh M(x - \xi)/l$.

On this basis a suitable quantitative evaluation with the purpose in view may be made of the integral terms on the right sides of Equations (4.2) and (4.3). Let us determine the distance $y = y_0$ from the left electrode beyond which these terms may be neglected with a practically sufficient degree of accuracy. In the case of flow in a rectangular channel with sides l and d it follows that for $d > 2y_0$ the function $f_0(\xi)$ which satisfies Equation (1.1), does not differ practically from $f_0(\xi)$, given by Equation (4.1).

For cross-sections of these proportions the influence of the right and the left electrodes on each other may be neglected. For $d < 2y_0$, excluding very small values of d, the function (4.1) may serve as a first approximation in finding $f_0(\xi)$.

5. In conclusion we shall show how by using the conformably transformed Green's function, we may arrive at a new form for the solution of Shercliff's problem for the flow of a conducting fluid through the same rectangular pipe where all walls are non-conductive.

Using the same basic equations as in Sections 1 and 2 of [2], and replacing only the boundary conditions (2.1) and (2.2) of [2] by the following

$$H_z|_{x=0} = H_z|_{x=l} = H_z|_{y=0} = H_z|_{y=d} = 0 \qquad (5.1)$$

we reduce the problem to the solution of Equations (2.12) of [3] for the boundary conditions

$$s|_{x=0} = -\beta e^{-\gamma a}, \qquad s|_{x=l} = \beta e^{\gamma a}$$
$$s|_{y=0} = s|_{y=d} = -\beta\left(1-\frac{x}{a}\right)e^{\gamma(x-a)} \qquad (5.2)$$
$$\left(\beta = \frac{Pa}{2\gamma\eta}\right)$$
$$t|_{x=0} = -\beta e^{\gamma a}, \qquad t|_{x=l} = \beta e^{-\gamma a}$$
$$t|_{y=0} = t|_{y=d} = -\beta\left(1-\frac{x}{a}\right)e^{-\gamma(x-a)} \qquad (5.3)$$

i.e. for boundary conditions of the first kind. Using the Green's function $G(\xi, \eta, x, y)$ for Equation (2.14) of [2], the function vanishes on the sides of the rectangle. It has the form

$$G(\xi, \eta, x, y) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \{K_0 (\gamma \sqrt{(x_m - \xi)^2 + (y_n - \eta)^2}) -$$
$$- K_0 (\gamma \sqrt{(x_m - \xi)^2 + (y_n' - \eta)^2}) - K_0 (\gamma \sqrt{(x_m' - \xi)^2 + (y_n' - \eta)^2}) +$$
$$+ K_0 (\gamma \sqrt{(x_m' - \xi)^2 + (y_n' - \eta)^2})\} \qquad (5.4)$$

(the notation here is the same as in (2.15 I), (2.16 I)). We find the value of s at any point inside the rectangle according to the equation

$$s = \int_0^d \{s|_{\xi=l} G_\xi' (l, \eta, x, y) - s_{\xi=0} G_\xi' (0, \eta, x, y)\} d\eta + \int_0^l s|_{\eta=0} [G_\eta' (\xi, 0, x, y) -$$
$$- G_\eta' (\xi, d, x, y)] d\xi = \beta \Big\{ e^{\gamma a} \int_0^d G_\xi' (l, \eta, x, y) d\eta + e^{-\gamma a} \int_0^d G_\xi (0, \eta, x, y) d\eta +$$
$$+ \frac{1}{a} \int_0^l (\xi - a) e^{\gamma(\xi - a)} [G_\eta' (\xi, 0, x, y) - G_\eta' (\xi, d, x, y] d\xi \Big\} \qquad (5.5)$$

and correspondingly for t. Thereupon u and v are obtained from the relations

$$u = \tfrac{1}{2} (p + q) = \tfrac{1}{2}[e^{-\gamma(x-a)} s + e^{\gamma(x-a)} t], \; v = \tfrac{1}{2} (p - q) = \tfrac{1}{2} [e^{-\gamma(x-a)} s - e^{\gamma(x-a)} t]$$

This form of the solution is particularly convenient in the case of large M.

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BIBLIOGRAPHY

1. Shercliff, I.A., Steady motion of conducting fluids in pipes under transverse magnetic fields. *Proc. of the Cambr. Phil. Soc.* No. 1, 49, 1953.

2. Grinberg, G.A., Ob ustanovivshemsia techenii provodiashchei zhidkosti v priamougol'noi trube s dvumia neprovodiashchimi stenkami i dvumia provodiashchimi, parallel'nymi vneshnemu magnitnomu poliu (On steady flow of a conducting fluid in a rectangular tube with two non-conducting walls and two conducting ones parallel to an external magnetic field). *PMM* Vol. 25, No. 6, 1961.

3. Akhiezer, N.I. and Shcherbina, V.A., Ob obrashchenii nekotorykh singuliarnykh integralov (On the conversion of some singular integrals). *Zapiski matem. otd. fiz.-matem. fak-ta Khar'kovskogo un-ta im. A.M. Gor'kogo i Khar'kovskogo matem. ob-va* Ser. 4, Vol. 25, 1957.

4. Slutskii, E.E., *Tablitsy dlia vychisleniia nepolnoi* Γ*-funktsii i funktsii veroiatnosti* χ^2 *(Tables for Calculation of Incomplete* Γ*-Function and Probability Function* χ^2). Izd-vo Akad. Nauk SSSR, 1950.

5. Pearson, K., *Tables of the Incomplete* Γ*-Function*. Printed by the Cambridge University Press and published by the Office of Biometrika, Re-issue, 1951.

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